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Original Research Article

Probability distribution function and constraints for the new entropy

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ABSTRACT
In this paper, a new method for determining a system's entropy is provided that yields a general form that is linearly dependent on component entropies, similar to extensive Renyi entropy but non-extensive like Tsallis entropy. This entropy has a conceptually new but straightforward origin and is easily defined mathematically by a fairly straightforward statement employing a derivative. It results in a probability distribution function that involves the Lambert function that is slightly more complex than the Shannon or Boltzmann forms but is still highly tractable mathematically and has never before been seen in this context. The author numerically compared it to the Tsallis and Shannon entropies.

1. Introduction

Entropy is a measurement of the disorder or randomness of a system with many constituents. It reaches its maximum value when a system can randomly exist in a number of states with equal probability, and it falls to zero at the lowest level when the system is in a single state with complete knowledge of its description. In addition to this characteristic that all definitions of entropy at the extremes have, differences are conceivable in the way the functional form in the middle is particularised [1]. For states with various energy or other conserved characteristics, they result in varied shapes of the probability distribution function. While some can be specified to be not so, others appear to be extensive, where the entropy of a combination of systems is just the sum of the entropies of the systems, as in the canonical instance of the Shannon form. Since Renyi [2] entropy differs from Shannon entropy while still being extensive, the extensibility attribute of the Shannon form is not unique.

In recent years, Tsallis entropy [3] has received a lot of attention due to its conceptual and theoretical novelty as well as the fact that it can be demonstrated in specific physical cases [5–11] to be the relevant form where non-extensivity is anticipated due to the interaction of the combined subsystems. It reduces to the normal Shannon entropy in the appropriate limiting instance, demonstrating the concept's consistency.

However, the author will bring entropy in this study from a fresh angle that will also resemble the typical form in the limit. We'll first outline the justification for this new definition before quickly contrasting it with the existing forms. The probability distribution for this entropy, which the author will henceforth, refer to as s-entropy, will then be determined. This entropy will be tied to the idea of rescaling the phase space.

The determination of the free energy is crucial to the evolution of a system's statistical mechanics since it is connected to the normalisation of the probability distribution function, which in turn regulates how all the ensemble's macroscopic features behave. Determining the free energy's value in a straightforward physical scenario in terms of temperature for Tsallis and the newer version will therefore involve first establishing a general prescription to obtain the free energy. The specific heat will then be obtained in both scenarios using it, and its variation with temperature will be examined.

2. Entropy

Consider a register with just one letter in it. Let p_i represent the set of probabilities for each of the N possible combinations of the letter A_i in this place. Although it is easily extended to states i of a single state of an ensemble where the individual systems can be in any N states with probability p_i, the author is employing the terminology of information theory in this context, as used, for example, in the Shannon Coding Theorem.

Let's now explore a slight resizing of the (single cell) register to make room for the letters $q = 1 + \Delta q$. The equivalent AND operation now determines the likelihood p_i^q that the letter p_i will fill the entire new phase space, and as a result, the likelihood that any of the pure letters A_i will fill the newly deformed cell is given by

$$N(q) = \sum_i p_i^q \,. \tag{1}$$

This would result in a decrease from the original total probability of unity for $q = 1$ for $q > 1$. It is clear that the deficit, which the author indicate by

$$M(q) = 1 - \sum_i p_i^q \,, \tag{2}$$

reflects the overall likelihood that the mixed cell contains a mixture of A_i and some other A_j, as the probability that the cell contains one or more letters overall (fractional included) must be one. As a result, the mixing probability $M(q)$ truly represents the degree of disorder that is brought about by changing the cell scale from unity to $1 + \Delta q$.

In a similar vein to establishing the fractal (Hausdorff) dimensions of dynamical attractors [12, 13] and in complex systems, fractional values of cell counts are introduced. Studies of diffusion [14] and percolation in complicated systems with fluids that effectively have fractional dimensions have been conducted. In these cases, the specific geometric limitations cause the associated space's dimension to change to a fractional dimension that is illogical on the surface. Huffmann coding, where the ideal alphabet size may formally be a fraction but may be modified for pragmatic reasons to the nearest higher integer, is one of the coding theories for the best transmission of information [15]. In order to accommodate a certain quantity of information, one can therefore consider a fractional size of the register or, alternatively, an integral number of cells in the register with fractionalsized cells. Therefore, our use of fractional cell sizes may be a classical precursor of the inevitable departure from stringent Shannon-type concepts. In quantum computing contexts, probabilistic optimization in place of the deterministic parameterization of classical Shannon information theory becomes inevitable.

Kaniadakis has also researched an intriguing alternate approach to the introduction of a parameter-dependent entropy and probability distribution [16, 17].

Let us define the entropy from the information content of the register by for an alphabet with m letters.

$$m^{S(q)\Delta q} = m^{(M(q+\Delta q)-M(q))} \tag{3}$$

As a result, the entropy shows that a tiny change in the register's cell size actually results in a change in the mixing likelihood. This results in:

$$S(q) = \frac{dM(q)}{dq} \,. \tag{4}$$

Otherwise put,

$$S(q) = \sum_i p_i^q \log p_i \,. \tag{5}$$

The Tsallis form and this differential form are similar, but they are not the same.

$$S_T(q) = -\sum_i \frac{(1-p_i^q)}{1-q} \,, \tag{6}$$

There appears to be a singularity in Eq. (6) when $q = 1$, which is the Shannon limit. If one expresses entropy as the expectation value of the (generalised or ordinary) logarithm, the distinction between the Tsallis expression and ours becomes more apparent:

$$S_T(q) = -\left\langle \log_q(p) \right\rangle \,, \tag{7}$$

where the generalized q-logarithm may be expressed as:

$$\log_q(p_i) = \frac{1-p_i^{q-1}}{1-q} \,. \tag{8}$$

In Ref. [18, 19], a type of entropy that like ours was presented:

$$S_{AD} = -\frac{\sum_i p_i^q \log p_i}{\sum_i p_i^q} \tag{9}$$

This adds an additional denominator term to standardise the weights for $\log p_i$.

From a different physical perspective, Wang [20] has also proposed a form that is comparable to our own and exploits the condition:

$$\sum_i p_i^q = 1 \,. \tag{10}$$

The definition of the expectation value using the straightforward probability distribution:

$$\left\langle O \right\rangle = \sum_i p_i O_i \,. \tag{11}$$

In the present example, the author maintains the standard logarithm while defining the expectation value with respect to the deformed probability corresponding to the extended cell.

$$S_s(q) = -\left\langle \log(p) \right\rangle_q \tag{12}$$

where

$$\left\langle O \right\rangle_q = \sum_i p_i^q O_i \,. \tag{13}$$

Due to the denominator sum in the Aczel-Daroczy form, the straightforwardness of the relationship between the weights and $\log p_i$ is lost. If one just redefine the probability in the Wang form as:

$$\tilde{p}_i = p_i^q \,, \tag{14}$$

then one obtains

$$S_W = -\frac{1}{q}\left\langle \log \tilde{p} \right\rangle \tag{15}$$

which is the standard Shannon form shrunk down. The Wang form is therefore broad, in contrast to our form, where the deformed probabilities do not add up to unity because one permits information leaks.

Tsallis entropy corresponds with Shannon entropy in the limit $q \rightarrow 1$, and as, one also obtain the normal Shannon entropy and $p_i^q \rightarrow p_i$. This is because the function $\log_q$ approaches the normal logarithm in this limit.

What constitutes the Renyi entropy is:

$$S_R(q) = \frac{\log \sum_i p_i^q}{1-q} . \qquad (16)$$

This entropy is comprehensive, meaning that it is simply additive for two subsystems for any value of q, much as Shannon entropy. One needs a slightly different expression of the extensibility axiom to obtain Shannon entropy uniquely:

$$S_{1+2} = S_1 + \sum_i p_{1i} S_2(i) , \qquad (17)$$

where, given that subsystem 1 is in state I, $S_2(i)$ is the entropy of subsystem 2.

3. Probability distribution for the new entropy

By maximising entropy under limitations, the p_i can be derived in terms of the energy of the states or perhaps also other criteria.

$$\sum_i p_i - 1 = 0 \qquad (18)$$

and

$$\sum_i p_i E_i - U - 0 . \qquad (19)$$

Let us now optimize with respect to the p_i, the constrained function given by

$$L = S + \beta\left(\sum_i p_i E_i - U\right) + \alpha\left(\sum_i p_i - 1\right), \qquad (20)$$

In Eq. (20), α and β stand for the Lagrange multipliers corresponding to the two constraints. One obtains

$$-\frac{q}{q-1} p_i^{-(q-1)} (\log p_i - 1) + 1 + \gamma p_i^{-(q-1)} = 0 , \qquad (21)$$

where the author has used for brevity $\gamma = \alpha + \beta E_i$.

The author takes into consideration the simpler form to relate the terms with p_i to the Lambert function.

$$-c \log y + dy = 1 \qquad (22)$$

that transforms into

$$-\frac{d}{c} y e^{-(dy/c)} = -\frac{d}{c} e^{1/c} \qquad (23)$$

which results

$$y = -\frac{c}{d} W\left(-\frac{d}{c} e^{1/c}\right) \qquad (24)$$

and using our parameters, one should have p_i,

$$p_i = \left[\frac{-qW(z)}{(\alpha + \beta E_i)(q-1)}\right]^{\frac{1}{1-q}} , \qquad (25)$$

in which

$$z = -e^{\frac{q-1}{q}} (\alpha + \beta E_i) \frac{q-1}{q} \qquad (26)$$

where the Lambert function as described by [21] is $W(z)$.

$$z = W(z) e^{W(z)} . \qquad (27)$$

In the Shannon situation, where the author obtain the Gibbs formula for p_i, the parameters α and β derived from the Lagrange's multipliers for the two constraints are related to the overall normalisation and to the relative scale of energy, i.e., to temperature ($1/(kT)$). In the Tsallis instance, p_i is represented by the well-known value as follows:

$$p_i = [\alpha + \beta(q-1)E_i]^{\frac{1}{1-q}} \qquad (28)$$

which, for $q \to 1$, is easily observed to decrease to Shannon form.

It can be demonstrated via mathematics that this form simplifies to the Shannon form for $q \to 1$.

By expanding, it is simple to see that Tsallis entropy is non-extensive.

$$\begin{aligned} S_{1+2}^T &= -\sum_{ij} \frac{p_i p_j (1 - p_i^q p_j^q)}{(1-q)^2} \\ &= S_1^T + S_2^T + (1-q) S_1^T S_2^T . \end{aligned} \qquad (29)$$

One should have a simple additive relation for Renyi entropy

$$S_{1+2}^R = S_1^R + S_2^R . \qquad (30)$$

Considering the new entropy,

$$S_{1+2}^S = S_1^S + S_2^S - M_2(q) S_1^S - M_1(q) S_2^S , \qquad (31)$$

where M_a refers to the subsystem a's (Eq. (2)) specified mixing probability of states.

4. Probability, Lambert function properties and constraints for the entropy

There are an infinite number of Riemann sheets divided by cuts in the transcendental equation that describes the Lambert function, and these cuts are connected to the cut of the log function from $-\infty$ to 0. A subscript n separates the several branch values for the same z, with n = 0 being the principal value and $W_0(z)$ being real along the real z-axis from $-1/e$ to ∞ (Figure 1).

One also gets another branch conventionally labelled $W_{-1}(z)$, which too gives real values for real z in the domain $-1/e < z < 0$, going down from -1 at $z = -1/e$ to $-\infty$ at $z = 0$.

One obtains four different regimes for the parameters:

(*a*) $\alpha + \beta E_i > 0$ and $q > 1$: A real positive p_i requires from [Eq. (25) and (26)] (with $\varepsilon \equiv q-1$ and $\gamma_i \equiv \alpha + \beta E_i$) that $W(z_i) < 0$, and hence

$$-1/\varepsilon < z_i < 0. \tag{32}$$

One also needs all p_i to be less than unity. Hence, one should have a cut-off value of E_i given by

$$|W(z_i)|\frac{q}{\varepsilon} \geq \alpha + \beta E_i\,, \tag{33}$$

where z_i is again dependent on E_i as given in Eq. (26) so that it is a transcendental equation.

(*b*) $\alpha + \beta E_i < 0$ and $q > 1$: Here, reality of p_i demands that $W(z_i) > 0$ so that initially z_i need only be positive. But as $\varepsilon < 0$ and is in the exponent, the condition $p_i < 1$ gives as a lower cut-off of z_i the value $\tilde{z}_i$, given by

$$W(\tilde{z}_i) = \frac{\gamma_i|\varepsilon|}{q} \tag{34}$$

which means that the negative E_i have a highest possible value given by Eq. (34), which is again a transcendental equation in E_i.

(*c*) $\alpha + \beta E_i > 0$ and $q < 1$: Reality of p_i implies in this case $W(z_i) > 0$ so that $z_i > 0$. The condition $p_i < 1$ similarly gives the cut-off $\tilde{z}_i$ defined by

$$W(\tilde{z}_i) = \frac{|\varepsilon|\gamma_i}{q}\,. \tag{35}$$

So, E_i has a maximum value given by this transcendental constraint.

(*d*) $\alpha + \beta E_i < 0$ and $q < 1$: In this case, $-1/e < z_i < 0$ initially due to reality of p_i and the constraint $p_i < 1$ leads to as in the previous cases to cut-off $\tilde{z}_i$ defined by

$$|W(\tilde{z}_i)| = \frac{|\varepsilon||\gamma_i|}{q} \tag{36}$$

and one can solve for the cut-off E_i from the other parameters numerically in specific problems with given sets of parameters.

The other branch of the Lambert function $W_{-1}(z)$ is also real and negative for $-1/_e < z < 0$ with the values -1 to $-\infty$, but is not acceptable as it does not give the limit $W_{-1}(z) \to 0$ as $z \to 0$, when $q \to 1$, which is required to get the Shannon limit for $q \to 0$.

Since an exponential has no finite root, Shannon entropy may be determined for any arbitrary number of energy because the probability function is of the exponential Boltzmann type. The author finds that when our entropy is utilised, the spectrum of energy levels E_i may be restricted for finitely non-zero $q - 1$. The functional dependency also exhibits a power behaviour and has finite roots in the case of Tsallis entropy.

5. Numerical analysis of the results

The author displays the change in the probability function for various E at various q values in Figure 2.

One may observe that the Shannon form is less curved than the new probability density function (pdf). The Gibbs exponential distribution is substantially different in form and size at high energy values, dropping more quickly at larger values of q. Even a 10% deviation from the norm for $q = 1$ can significantly alter the pdf, which should be fairly simple to observe in experimental settings. The form is essentially linear when $q = 1.3$.

One may compare Tsallis' pdf and the pdf for the new entropy for the identical values of q, 1.1 in the former and 1.3 in the latter, in Figures 3 and 4. One may observe that the new entropy provides much stiffer probability curves for bigger q values, diverging significantly from the Tsallis' pdf.

One may observe that, unlike Shannon's exponential form, which the author discussed in Section 4, the pdf for both the Tsallis form and our new form of entropy hit the axis at finite values of the energy, making the support of the probability finite.

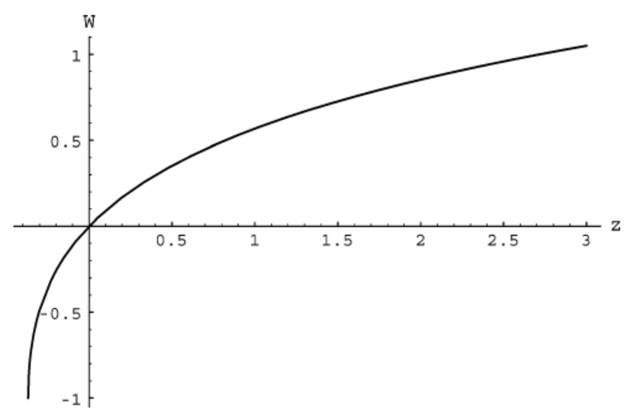

Figure 1. The $W_0(z)$ branch, which is real from $z = 1/e$ to $z = 0$, is not shown since, as stated in the text, it is not appropriate for our entropy. $W_0(z)$ is real along the real axis from $1/e$ to ∞; the value of W_0 varies from -1 to ∞.

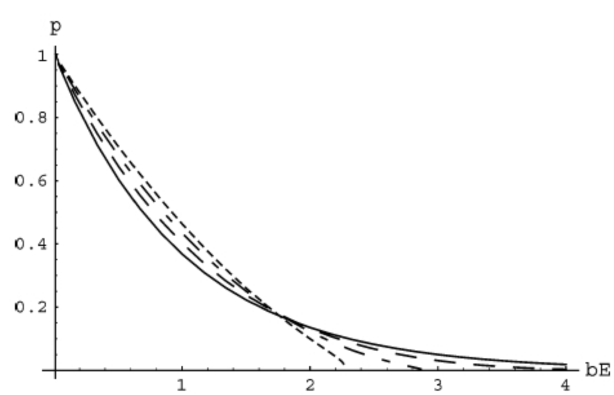

Figure 2. Comparison of the new entropy's PDF for $q = 1$, 1.1, 1.2, and 1.3 values. The Gibbs exponential distribution, represented by the solid line for $q = 1$, has lines in the order of q.

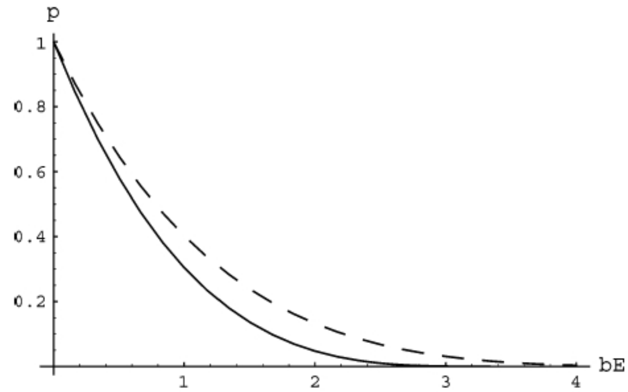

Figure 3. Comparison of the new entropy presented here for q = 1.1 and the Tsallis non-extensive entropy (solid line).

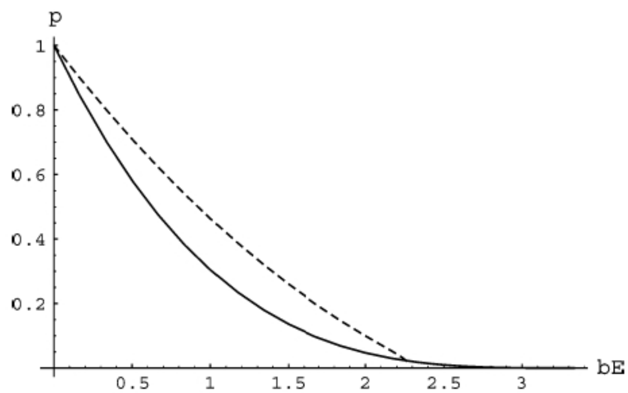

Figure 4. Same as Fig. 3, but for a higher q = 1.3.

6. Conclusions

One can see that the novel entropy proposed here, which is based on the straightforward idea of the quantity of state mixing freedom introduced per unit cell of phase space, results in a non-extensive form that is distinct from all other entropies currently under study. As a result, the pdf takes on a complex yet integrable shape that significantly deviates from Tsallis entropy. In a different way than Tsallis entropy, this entropy is also non-extensive, although as is to be expected, it too extends trivially in the limit $q \to 1$. In contrast to the classical exponential form of the pdf, which has an infinite energy support, one should have the interesting deviation that our form, like that of Tsallis, leads to a finite energy support. As a result, in theory, values of the energy may be ruled out for ranges of parameters, such as temperature, that make the probability unphysical.

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