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Original Research Article

Some methods for determination of entropy of decreasing, increasing and hybrid function of variables

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ABSTRACT
A topic of mathematics that studies subjective uncertainty is known as uncertainty theory. Entropy is a crucial idea that offers a numerical assessment of the uncertainty connected to uncertain variables. This study suggests a few entropy formulas for functions of unknown variables with regular uncertain distributions to facilitate the computation of entropy. Finally, directly following from these formulations is the proof of the entropy operator's positive linearity feature.

1. Introduction

An objective way to quantify the level of uncertainty is through entropy. Zadeh [2] was the first to use fuzzy entropy, which he described as a weighted Shannon entropy, to measure fuzziness. He was inspired by the Shannon entropy of random variables [1]. Numerous researchers, including De Luca and Termini [3], Kaufmann [4], Yager [5], Kosko [6], Pal and Pal [7], Pal and Bezdek [8], Bhandari, Pal [9], and Liu [10], have researched fuzzy entropy to date. These definitions of entropy, however, described the uncertainty caused by the challenge of determining whether or not an element belongs to a set, i.e., they characterised the uncertainty caused by linguistic vagueness rather than information deficiency, and they disappeared when the fuzzy variable is an equipossible one. According to Liu [11], an entropy should adhere to the following three criteria. Minimum: A crisp integer has a minimum entropy of 0, or 0. Entropy of an equipossible fuzzy variable is at its highest level. Universality: The entropy is relevant in both discrete and continuous circumstances, in addition to finite and infinite cases. Li and Liu [12] developed a new definition of fuzzy entropy based on these specifications to describe the uncertainty brought on by a lack of knowledge.

In 2007, Liu [13] proposed the idea of uncertain measure as a crucial component of uncertainty theory, which is now being developed as a discipline of mathematics that investigates uncertainty in human systems. It satisfies some mathematical qualities including normalcy, monotonicity, self-duality, and countable subadditivity and is essentially a set function. The application of uncertainty theory to uncertain processes, uncertain processes, uncertain differential equations, uncertain optimum controls, and other areas has grown in recent years. Gao [14] has researched the semi-canonical process' characteristics.

In order to describe the uncertainty of uncertain variables brought on by a lack of information, Liu [15] proposed the idea of entropy of uncertain variables in 2009. There are an endless number of uncertainty distributions that are consistent with the existing knowledge in many practical situations when only partial information is available about uncertain variables like anticipated value and variance. According to the maximum entropy principle, Jaynes [16] advised selecting the distribution for random variables that has the highest entropy. The maximum entropy concept of uncertainty distribution for uncertain variables was studied by Chen and Dai [17].

Entropy of unknown variables is a crucial idea that is frequently employed in both theory and practise. However, using the definition alone to directly determine the entropy of a function of uncertain variables is still rather challenging. The author derived several formulas to determine the entropies of functions of uncertain variables with regular distribution in order to improve the entropy calculation algorithm. To demonstrate how to apply these formulas, some examples are provided. By using these formulas, the author demonstrated in this study that the entropy operator has a positive scalability property. It might be useful in various operation research issues where the entropy is used in the modelling process.

2. Some basic concepts

An uncertain measure M is a real-valued set-function on a σ-algebra P over a nonempty set T. Each element in P is called an event. The value $M\{A\}$ is a number assigned to event A, which indicates the uncertainty that A will occur. An uncertain measure is normal, self-dual and countable subadditivity.

Axiom 1 (*Normality*). $M\{\Gamma\} = 1$.

Axiom 2 (*Self-Duality*). $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for event Λ.

Axiom 3 (*Countable Subadditivity*). For every countable sequence of events $\{\Lambda_i\}$, one should have

$$M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

Axiom 4 (*Product Measure Axiom*). Let Γ_k be non-empty sets on which M_k are uncertain measures, $k = 1, 2, \ldots, n$, respectively. Then the product uncertain measure M is an uncertain measure on the product σ-algebra $P = P_1 \times P_2 \times \ldots \times P_n$ satisfying

$$M\left\{\prod_{i=1}^{n}\Lambda_k\right\} = \min_{1\le i\le n} M_k\{\Lambda_k\}.$$

Definition 1 (Liu [13]). Let T be a nonempty set, P a σ-algebra over T, and M an uncertain measure. Then the triplet (T, P, M) is called an uncertainty space.

Definition 2 (Liu [13]). An uncertain variable is a function from an uncertainty space (T, P, M) to the set of real numbers.

You [18] and Gao [19] studied some properties of uncertain measure. The uncertainty distribution function $\Phi : R \to [0, 1]$ of an uncertain variable ξ is defined as $\Phi(x) = M\{\xi \le x\}$. It has been proved by Peng and Iwamura [20] that a function is an uncertainty distribution function if and only if it is an increasing function except $\Phi(x) = 0$ and $\Phi(x) = 1$.

Definition 3 (Liu [13]). The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$M\left\{\bigcap_{i=1}^{m}\xi_i \in B_i\right\} = \min_{1\le i\le m} M\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

Definition 4 (Liu [13]). Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \ge r\}dr - \int_{-\infty}^{0} M\{\xi \le r\}dr$$

provided that at least one of the two integrals is finite.

The regular uncertain variable is discussed in the final definition. If an uncertain variable has a regular uncertainty distribution, it is referred to as a regular uncertain variable.

Definition 5 (Liu [11]). An uncertainty distribution Φ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.

The distribution of functions for uncertain variables is then deduced using three theorems that are introduced in Liu [11].

Theorem 1 (Liu [11]). Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f : R_n \to \mathrm{R}$ is a strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

is an uncertain variable with uncertainty distribution

$$\psi(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1\le i\le n}(\Phi_i(x_i)),\ x \in R$$

whose inverse function is

$$\psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)),\ 0 < \alpha < 1.$$

Theorem 2 (Liu [11]). Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f : R_n \to \mathrm{R}$ is a strictly decreasing function, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

is an uncertain variable with uncertainty distribution

$$\psi(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1\le i\le n}(1 - \Phi_i(x_i)),\ x \in R$$

and its inverse function may be expressed as:

$$\psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)),\ 0 < \alpha < 1.$$

Theorem 3 (Liu [11]). Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f : R_n \to R$ is a strictly increasing function with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing function with $x_{m+1}, x_{m+2}, \ldots, x_n$, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

is an uncertain variable with uncertainty distribution

$$\psi(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1\le i\le m} \Phi_i \wedge \min_{m+1\le i\le n}(1 - \Phi_i(x_i)),\ x \in R$$

and its inverse function may be expressed as:

$$\psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)),$$
$$0 < \alpha < 1.$$

3. Entropy theorems

Entropy is a notion that will be introduced in this part, along with many formulas that may be used to determine its value for functions of uncertain variables with regular distributions. The following definition applies to Liu's [15] concept of entropy.

Definition 6 (Liu [15]). Suppose that ξ is an uncertain variable. Then its entropy is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} S(M\{\xi \le x\})dx,$$

where $S(t) = -t \ln t - (1 - t) \ln (1 - t)$.

Theorem 4. Assume ξ is an uncertain variable with regular uncertainty distribution Φ. If the entropy $H[\xi]$ exists, then

$$H[\xi] = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha .$$

Proof. Since ξ has a regular uncertainty distribution Φ, one should have

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x)dx$$

$$= \int_{-\infty}^{0} \int_{0}^{\Phi(x)} S^{'}(\alpha) d\alpha dx + \int_{0}^{\infty} \int_{\Phi(x)}^{1} -S^{'}(\alpha) d\alpha dx$$

where $S^{'}(\alpha) = (-\alpha \ln \alpha - (1-\alpha)\ln(1-\alpha))^{'} = -\ln \frac{\alpha}{1-\alpha}$

By the Fubini theorem, one should have

$$H[\xi] = \int_{0}^{\Phi(0)} \int_{\Phi^{-1}(\alpha)}^{0} S^{'}(\alpha) d\alpha dx + \int_{\Phi(0)}^{1} \int_{0}^{\Phi(\alpha)} -S^{'}(\alpha) d\alpha dx$$

$$= -\int_0^1 \Phi^{-1}(\alpha) S^{'}(\alpha) d\alpha$$

$$= \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha .$$

The theorem is proved.

Theorem 5. Assume ξ_1, ξ_2, ….., ξ_n are independent uncertain variables with regular uncertainty distribution Φ_1, Φ_2, ….. , Φ_n, respectively. If $f : R_n \rightarrow R$ is a strictly monotone function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an entropy

$$H[\xi] = \left| \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots\ldots, \Phi_n^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha \right| .$$

Proof. If f is a strictly increasing function, letting ψ be the distribution function of $f(\xi)$, it easily follows from Theorem 1 that

$$\psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots\ldots, \Phi_n^{-1}(\alpha)) .$$

Then Theorem 4 demonstrates that

$$H[f(\xi)] = \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots\ldots, \Phi_n^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha .$$

Otherwise, if f is a strictly decreasing function, by Theorem 2, it turns that

$$\psi^{-1}(\alpha) = f(\Phi^{-1}(1-\alpha), \Phi^{-1}(1-\alpha), \ldots\ldots, \Phi^{-1}(1-\alpha)) .$$

Then Theorem 4 demonstrates that

$$H[f(\xi)] = \int_0^1 f(\Phi^{-1}(1-\alpha), \Phi^{-1}(1-\alpha), \ldots\ldots, \Phi^{-1}(1-\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha$$

$$= -\int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots\ldots, \Phi_n^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha .$$

$$H[\xi] = \left| \int_0^1 f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots\ldots, \Phi_n^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha \right| .$$

The theorem is proved.

Theorem 6. Assume ξ_1, ξ_2, ….., ξ_n are independent uncertain variables with regular uncertainty distribution Φ_1, Φ_2, ….., Φ_n, respectively. If $f : R_n \rightarrow R$ is a strictly increasing function with respect to x_1, x_2, ….., x_m and strictly decreasing function with x_{m+1}, x_{m+2}, ….., x_n, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an entropy

$$H[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \ldots\ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots\ldots, \Phi_n^{-1}(1-\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha$$

Proof. Let ψ be the distribution function of $f(\xi)$. Then Theorem 3 demonstrates that

$$\psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots\ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots \Phi_n^{-1}(1-\alpha)) .$$

Then Theorem 4 demonstrates that

$$H[f(\xi)] = \int_0^1 f(\Phi_1^{-1}(\alpha), \ldots\ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots\ldots, \Phi_n^{-1}(1-\alpha)) \ln \frac{\alpha}{1-\alpha} d\alpha$$

The theorem is proved.

4. Entropy operator

In this section, the author shall prove the positive linearity of entropy.

Theorem 7. Let ξ and η be independent uncertain variables. Then for any real numbers a and b, one should have

$$H[a\xi + b\eta] = |a| H[\xi] + |b| H[\eta].$$

Proof. Suppose that ξ and η are independent uncertain variables with regular distributions Φ and ψ, respectively. Otherwise, one may give them a small perturbation such that the uncertainty distributions are regular. Theorem 5 demonstrates that

$$H[\xi]=\int_0^1 (\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}d\alpha,$$

$$H[\eta]=\int_0^1 (\psi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}d\alpha.$$

Since the function $f(x, y) = x + y$ is strictly monotone, Theorem 5 demonstrates that

$$H[\xi+\eta]=\int_0^1 (\Phi^{-1}(\alpha)+\psi^{-1}(\alpha))\ln\frac{\alpha}{1-\alpha}d\alpha$$

$$=H[\xi]+H[\eta].$$

One proves $H[a\xi] = aH[\xi]$. If $a = 0$, then the equation holds trivially. If $a > 0$, Theorem 5 demonstrates that

$$H[a\xi]=\left|\int_0^1 a\Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}d\alpha\right|=|a|H[\xi].$$

Finally, for any real numbers a and b, above theorem demonstrates that

$$H[a\xi + b\eta] = H[a\xi] + H[b\eta] = |a|H[\xi] + |b|H[\eta].$$

The theorem is proved.

5. Conclusions

In this article, the author discussed a few techniques for calculating the entropy of functions with regular distributions and uncertain variables. The author provided a number of formulas for rising, decreasing, and hybrid functions to simplify the calculation. This paper concludes by proving the entropy operator's positive linearity feature.

References

[1] C. Shannon, The Mathematical Theory of Communication, The University of Illinois Press, Urbana (1949).

[2] L. Zadeh, Probability measures of fuzzy events, *J. Math. Anal. Appl.* **23** (1968) 421-427.

[3] A. De Luca, S. Termini, A definition of nonprobabilitistic entropy in the setting of fuzzy sets theory, *Info. Control* **20** (1972) 301-312.

[4] A. Kaufmann, Introduction to the Theory of Fuzzy Subsets, Academic Press, New York (1975).

[5] R. Yager, On measures of fuzziness and negation, part I: membership in the unit interval, *Int. J. Gen. Syst*. **5** (1979) 221-229.

[6] B. Kosko, Fuzzy entropy and conditioning, *Info. Sci*. **40** (1986) 165-174.

[7] N. Pal, K. Pal, Object background segmentation using a new definition of entropy, *IEE Proc*. **136** (1989) 284-295.

[8] N. Pal, J. Bezdek, Measuring fuzzy uncertainty, *IEEE Trans. Fuzzy Syst*. **2** (1994) 107-118.

[9] D. Bhandary, N. Pal, Some new information measures for fuzzy sets, *Info. Sci*. **67** (1993) 209-228.

[10] X. Liu, Parameterized defuzzification with maximum entropy weighting function–another view of the weighting function expectation method, *Math. Comp. Model*. **45** (2007) 177-188.

[11] B. Liu, Asurvey of entropy of fuzzy variables, *J. Uncert. Syst*. **1** (2007) 4-13.

[12] P. Li, B. Liu, Entropy of credibility distributions for fuzzy variables, *IEEE Trans. Fuzzy Syst*. **16** (2008) 123-129.

[13] B. Liu, Uncertainty Theory, Springer-Verlag, Berlin (2007).

[14] Y. Gao, Variation analysis of semi-canonical process, *Math. Comp. Model*. **53** (2011) 1983-1989.

[15] B. Liu, Some research problems in uncertainty theory, *J. Uncert. Syst*. **3** (2009) 3-10.

[16] E. Jaynes, Information theory and statistical mechanics, *Phys. Rev*. **106** (1957) 620-630.

[17] X. Chen, W. Dai, Maximum entropy principle for uncertain variables, *Int. J. Fuzzy Syst*. **13** (2011) 234.

[18] C. You, Some convergence theorems of uncertain sequences, *Math. Comp. Model*. **49** (2009) 482-487.

[19] X. Gao, Some properties of continuous uncertain measure, *Int. J. Uncert*., **17** (2009) 419-426.

[20] Z. Peng, K. Iwamura, Uncertainty distribution of functions of uncertain variables, *J. Interdisc. Math*. **13** (2010) 277-285.

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